A generalized approach to local regularization of linear Volterra problems in L^p spaces

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Abstract. A generalized version of local regularization is developed. The convergence criteria set forth in the generalization are shown to be realized naturally by a particular local regularization scheme when applied to finitely-smoothing linear Volterra convolution equations in the Banach spaces $L^p(0,1), 1 \leq p \leq \infty$. The method leads to convergence with a priori parameter selection for $1 < p < \infty$ and under assumptions of increased regularity of the true solution for $1 \leq p \leq \infty$. Rates of convergence are established beyond those previously known for local regularization of this problem in $C[0,1]$ and under more general source conditions. Numerical examples are included to illustrate implementation and effectiveness of the method.

1. Introduction

We consider the inverse problem of solving

$$\mathcal{A}u = f, \tag{1}$$

for $\bar{u} \in X$, where X is any Banach space of functions on the bounded domain $\Omega \subseteq \mathbb{R}$. Here $\mathcal{A} : X \mapsto X$ is a compact linear operator, injective with nonclosed range, and $f \in \mathcal{R}(\mathcal{A}) \subseteq X$ denotes "exact" data for the equation.

It follows from our assumptions that the solution to (1) does not depend continuously on data f. A regularization method must therefore be used in order to provide a reasonable reconstruction of $\bar{u}$ in the usual situation that f is only available approximately. Let $f^\delta \in X$ denote the measured or inexact data, and assume throughout that

$$\left\| f^\delta - f \right\| < \delta, \tag{2}$$

for some $\delta > 0$ small.

With a regularization method one uses the measured data f^δ to construct an approximate solution u_α^δ to a nearby stable problem defined by the method. The scalar $\alpha > 0$ is the regularization parameter which controls the stability and proximity to the original problem. Convergent methods are those for which choices of $\alpha = \alpha(\delta) > 0$ are possible which guarantee that $u^\delta_{\alpha(\delta)} \to \bar{u}$ as $\delta \to 0$.

As an example, we focus here on the case where $X = L^p(0,1), 1 \le p \le \infty$ with usual norm $\|\cdot\|$ and $\mathcal{A} \in \mathcal{L}(X)$ is the Volterra convolution operator given by

$$\mathcal{A}u(t) := \int_0^t k(t-s)u(s)\,ds, \qquad \text{a.e. } t \in (0,1), \tag{3}$$

with ν-smoothing kernel $k \in C^\nu[0,1]$ for some $\nu \ge 1$, i.e.,

$$k^{(\ell)}(0) = 0, \quad \ell = 0,1,...,\nu-2, \qquad \text{and} \qquad k^{(\nu-1)}(0) \neq 0.$$

We refer to $\mathcal{A}$ defined in (3) as a *ν–smoothing operator* and in the following refer to (1) as the *ν–smoothing Volterra problem* on X when $\mathcal{A}$ is ν-smoothing. In this case, the operator $\mathcal{A}$ is clearly compact and injective with nonclosed range. Without loss of generality we assume that for the ν-smoothing problem, equation (1) is scaled so that $k^{(\nu-1)}(0) = 1$.

The ν–smoothing Volterra problem is a generalization of the problem of obtaining the ν-th order derivative of f, corresponding to the kernel $k(t) = t^{\nu-1}/(\nu-1)!$, where $f(0) = f'(0) = \cdots = f^{(\nu-1)}(0) = 0$. Other practical examples include the one-smoothing problem of determining a population's propagation rate from measurements of total population [23], and the two-smoothing problem of determining the density of a chain given information about its motion as it slides down the surface of a cycloid [7].

1.1. Regularization methods for Volterra problems

In general, classical methods such as Tikhonov regularization are deemed unsuitable for Volterra problems due to the fact that the causal structure of the original problem is lost ([13, 19]). In the Hilbert space setting such methods require use of the adjoint $\mathcal{A}^*$ of the operator $\mathcal{A}$ leading to regularized equations that are no longer Volterra, e.g. Tikhonov regularization applied to the ν-smoothing problem leads to solving a Fredholm equation.

This loss is also evident when the regularized equation is discretized, leading to less efficient numerical implementation. However when methods such as Lavrent'ev regularization and *local regularization* are applied to the ν-smoothing problem, the regularized equation is still Volterra, preserving the causal nature of the problem and leading to faster and more efficient numerical realizations which can be solved in a sequential manner. For a discussion and comparison of other methods which preserve the structure of a Volterra problem, see [1, 13] and the references therein.

In the 1960's, J.V. Beck developed a method in [2] for approximating the solution to the discretized inverse heat conduction problem, an approach which continues to be used successfully in practice. In the 1990's, convergence and stability of the method was established in [12], followed by a generalization to the continuous case. This work resulted in the development of a class of regularization methods referred to as *sequential predictor-corrector regularization*, *future-sequential regularization*, or just *local regularization* for which Beck's method is a special case. We refer the reader to [11, 13–17, 20, 21] for developments of the theory for linear problems and [5, 8, 16] for developments of the nonlinear theory to date.

Local regularization with *a priori* parameter selection is a convergent method for the ν-smoothing Volterra problem in the space $C[0,1]$ and a rate of uniform convergence is known under the assumption of Hölder continuity [15]. An *a posteriori* parameter selection strategy is proposed in [3, 4]. A convergence theory for the underlying data space $L^p(0,1), 1 \leq p \leq \infty$ and improved rates of convergence are provided in this paper.

For the special case of a ν-smoothing Volterra operator $\mathcal{A}$ in (3), it is well-known that additional data is needed to accurately reconstruct the solution $\bar{u}$ on the entire interval $(0,1)$ (see e.g., [15, 21]). To handle this situation, either additional data is required beyond $(0,1)$ or else one settles for a reconstruction of the solution on a slightly smaller interval $(0, 1-\alpha)$, for some $\alpha > 0$ small. In the references listed previously, the former case is assumed. While the general theory we develop handles this case, we address here the latter case by approximating $\bar{u} \in X = L^p(0,1)$ by a suitable reconstruction u_α^δ in the space X_α, where $X_\alpha = L^p(0, 1-\alpha)$, for $\alpha > 0$ small.

The organization of the paper is as follows. We begin by formulating a generalization of the method of local regularization where for each $\alpha > 0$, the solution of an equation of the form

$$a_\alpha u + \mathcal{A}_\alpha u = T_\alpha f^\delta$$

is used as a regularized approximation to the solution of (1). A detailed construction of the generalized method is given in Section 2.

In Section 3, we demonstrate how the conditions set forth in the generalization are realized naturally by a local regularization method designed to regularize the ν-smoothing problem. The method we introduce is the standard theory of (0^{th} order) local regularization, (established for this problem in the case of $X = C[0,1]$ in [12, 14, 15, 22], etc., and in [21] with a first order local method), however extended here to include the more general Lebesgue spaces $X = L^p(0,1)$, $1 \leq p \leq \infty$. In addition, we obtain improved rates of convergence under general source conditions beyond those previously known.

We conclude with some numerical examples in Section 4. A comparison of effectiveness is made among Tikhonov regularization, the method of Lavrent'ev, and the generalized method of local regularization in the reconstruction of a piecewise linear function. This is followed by an example which illustrates the construction of measures associated with local regularization methods.

2. A generalized approach to local regularization

2.1. Motivation

Before giving precise definitions and assumptions, we motivate some of the ideas behind local regularization. Let $\alpha > 0$ be a small fixed parameter. Suppose we are given a data sampling operator T_α, a bounded linear operator which is defined on the original Banach space X of functions on the bounded domain $\Omega \subseteq \mathbb{R}$ and which may have its range in another Banach space X_α of functions on $\Omega_\alpha \subseteq \Omega$. We first apply the operator to both sides of (1) to obtain

$$T_\alpha \mathcal{A} u = T_\alpha f, \tag{4}$$

which is still an ill-posed equation. One might now choose to regularize via the addition of a stability term αu, to the left-hand side of (4), leading to the regularization equation

$$\alpha u + T_\alpha \mathcal{A} u = T_\alpha f^\delta, \tag{5}$$

in the case of noisy data f^δ.

Remark 2.1. *Note that when X is a Hilbert space, Tikhonov regularization and Lavrent'ev regularization have approximating equations of the form (5) for the choice of sampling operators $T_\alpha = \mathcal{A}^*$ and $T_\alpha = I$, the identity operator on X, respectively, and $X_\alpha = X$ for both methods.*

The method of local regularization of Volterra problems on $\Omega = (0,1)$ typically involves a data sampling operator $T_\alpha \in \mathcal{L}(X, X_\alpha)$ of the form

$$T_\alpha f^\delta(t) = \frac{\int_0^\alpha f^\delta(t+\rho)\, d\eta_\alpha(\rho)}{\int_0^\alpha d\eta_\alpha(\rho)}, \qquad \text{a.e. } t \in \Omega_\alpha = (0, 1-\alpha), \tag{6}$$

the η_α–integral average of f^δ over the interval $(t, t+\alpha)$ for a specified Borel measure η_α. Thus in local regularization, $T_\alpha f^\delta(t)$ provides a weighted "average" of the part of f^δ on $(t, t+\alpha)$ known to be useful in the reconstruction of $\bar{u}$ at t (see e.g. [2, 12]).

Local regularization involves an equation with structure similar to that of (5) but with slightly different terms on the left-hand side of that equation. We decompose $T_\alpha \mathcal{A}$ as

$$T_\alpha \mathcal{A} = D_\alpha + (T_\alpha \mathcal{A} - D_\alpha),$$

for an operator D_α which is "nearly diagonal" on the true solution $\bar{u}$, and regularize by replacing D_α with $a_\alpha I$, so that $D_\alpha \bar{u} \approx a_\alpha \bar{u}$ in a suitable sense for some scalar $a_\alpha \neq 0$. With this decomposition, the unregularized equation (4) can be written as

$$D_\alpha u + (T_\alpha \mathcal{A} - D_\alpha) u = T_\alpha f,$$

and the regularized form of the equation is

$$a_\alpha u + (T_\alpha \mathcal{A} - D_\alpha) u = T_\alpha f^\delta$$

in the case of noisy data. That is, in contrast to the methods of Tikhonov and Lavrent'ev, the stability term $a_\alpha u$ arises from a decomposition of $T_\alpha \mathcal{A}$.

2.2. Precise formulation of the method

Let $\bar{\alpha} > 0$ and let $\alpha \in (0, \bar{\alpha}]$ be a regularization parameter. The basic definitions and assumptions for a generalized version of local regularization are stated below.

A1. Let $[X_\alpha, r_\alpha]$ denote the pairing of a Banach space $(X_\alpha, \|\cdot\|_\alpha)$ and a well-defined linear operator $r_\alpha : X \mapsto X_\alpha$ which serves to facilitate movement between the two spaces.

A2. Let the "data sampling" operator $T_\alpha \in \mathcal{L}(X, X_\alpha)$ satisfy

$$\|T_\alpha g\|_\alpha \leq M_T \|g\|, \quad g \in X,$$

for $M_T > 0$ independent of $\alpha \in (0, \bar{\alpha}]$.

A3. The operator $T_\alpha \mathcal{A}$ may be written

$$T_\alpha \mathcal{A} = D_\alpha + \mathcal{A}_\alpha r_\alpha,$$

for $D_\alpha \in \mathcal{L}(X, X_\alpha)$ and $\mathcal{A}_\alpha \in \mathcal{L}(X_\alpha)$, where, for some $a_\alpha \neq 0$ the following is true.

(i) The operator $(a_\alpha + \mathcal{A}_\alpha)$ has a bounded inverse on X_α, with operator norm satisfying

$$\left\|(a_\alpha + \mathcal{A}_\alpha)^{-1}\right\|_{\mathcal{L}(X_\alpha)} \leq \frac{1}{c(\alpha)}, \tag{7}$$

for some $c(\alpha) > 0$.

(ii) The operator D_α is approximated by $a_\alpha r_\alpha$ on $\bar{u}$ in the sense that

$$\|(D_\alpha - a_\alpha r_\alpha)\,\bar{u}\,\|_\alpha = o(c(\alpha)) \quad \text{as} \ \ \alpha \to 0^+, \tag{8}$$

where $c(\alpha)$ given in **A3**(i) is assumed to satisfy

$$c(\alpha) \to 0, \quad \text{as} \ \ \alpha \to 0^+. \tag{9}$$

The generalized local regularization equation is

$$(a_\alpha + \mathcal{A}_\alpha)\, u = T_\alpha f^\delta. \tag{10}$$

Remark 2.2. *One anticipates that the boundedness of $(a_\alpha + \mathcal{A}_\alpha)^{-1}$ weakens as α approaches zero so that the scalar $c(\alpha)$ defined in (7) is reasonably expected to satisfy (9).*

Remark 2.3. *The methods of Tikhonov regularization and Lavrent'ev regularization can be shown to satisfy assumptions* **A1** *and* **A2** *with the choices of T_α and X_α given in Remark 2.1, and $r_\alpha = I$. With the trivial splitting $D_\alpha = \boldsymbol{O}$, the zero operator on X, and $a_\alpha = \alpha$, the approximations generated by Tikhonov regularization, $u_{tik} = (\alpha + \mathcal{A}^*\mathcal{A})^{-1}\mathcal{A}^* f^\delta$, and the method of Lavrent'ev (e.g., for non-negative, self-adjoint operators $\mathcal{A}$), $u_{lav} = (\alpha + \mathcal{A})^{-1} f^\delta$, are solutions of their respective versions of equation (10). However for both of these methods, $c(\alpha) = \alpha$ and*

$$\|(D_\alpha - a_\alpha)\bar{u}\| = \alpha \|\bar{u}\|,$$

and thus **A3***(ii) fails to hold unless $\bar{u} = \mathbf{0}$.*

A convergence theory for the generalized version of local regularization outlined above (with *a priori* parameter selection) is a straightforward consequence of assumptions **A1**–**A3**.

Theorem 2.1. *Assume that* **A1**–**A3** *hold for all $\alpha \in (0, \bar{\alpha}]$. Then for each such α and for any $f^\delta \in X$, there exists a unique solution $u_\alpha^\delta \in X_\alpha$ of the generalized local regularization equation (10) which depends continuously on data $f^\delta \in X$. Further, if*

$\|f^\delta - f\| \le \delta$, *for* $\delta > 0$, *and if any selection of the parameter* $\alpha = \alpha(\delta) \in (0, \bar{\alpha}]$ *is made satisfying*

$$\alpha(\delta) \to 0 \quad \text{and} \quad \frac{\delta}{c(\alpha(\delta))} \to 0 \quad \text{as} \ \ \delta \to 0 \tag{11}$$

for $c(\alpha)$ *given in* **A3**, *it follows that*

$$\|u^\delta_{\alpha(\delta)} - r_{\alpha(\delta)}\bar{u}\|_{\alpha(\delta)} \to 0 \quad \text{as} \ \ \delta \to 0. \tag{12}$$

Proof. For $\alpha \in (0, \bar{\alpha}]$, it is clear that $u^\delta_\alpha \in X_\alpha$ given by

$$u^\delta_\alpha = (a_\alpha + \mathcal{A}_\alpha)^{-1} T_\alpha f^\delta, \tag{13}$$

uniquely solves (10) and that the continuity of the mapping $f^\delta \in X \mapsto u^\delta_\alpha \in X_\alpha$ follows from **A2** and **A3**(i) via the relationship

$$\left\|u^\delta_\alpha\right\|_\alpha \le \frac{M_T}{c(\alpha)} \left\|f^\delta\right\|.$$

Further, since $T_\alpha \mathcal{A}\bar{u} = T_\alpha f$, we have

$$(a_\alpha + \mathcal{A}_\alpha)\, r_\alpha \bar{u} = T_\alpha f - (D_\alpha - a_\alpha r_\alpha)\, \bar{u}. \tag{14}$$

Subtracting equation (14) from equation (10) replacing f^δ with f and $u = u_\alpha$ leads to

$$\begin{aligned} \|u_\alpha - r_\alpha \bar{u}\|_\alpha &= \left\|(a_\alpha + \mathcal{A}_\alpha)^{-1} (D_\alpha - a_\alpha r_\alpha)\, \bar{u}\right\|_\alpha \\ &\le \left\|(a_\alpha + \mathcal{A}_\alpha)^{-1}\right\| \, \|(D_\alpha - a_\alpha r_\alpha)\, \bar{u}\|_\alpha \\ &\le \mathcal{O}\left(1/c(\alpha)\right) \cdot o(\, c(\alpha))\,) \to 0, \ \text{as} \ \alpha \to 0 \end{aligned} \tag{15}$$

by assumption **A3**. Subtracting equation (14) from equation (10) once again, but this time with $u = u^\delta_\alpha$ leads to

$$\begin{aligned} \left\|u^\delta_\alpha - r_\alpha\, \bar{u}\right\|_\alpha &= \left\|(a_\alpha + \mathcal{A}_\alpha)^{-1} \left[T_\alpha(f^\delta - f) + (D_\alpha - a_\alpha r_\alpha)\, \bar{u}\right]\right\|_\alpha \\ &\le M_T \frac{\delta}{c(\alpha)} + \frac{\|(D_\alpha - a_\alpha r_\alpha)\, \bar{u}\|_\alpha}{c(\alpha)} \end{aligned}$$

so that for $\alpha(\delta)$ chosen according to (11), the convergence in (12) is obtained. □

Under assumptions of increased regularity on $\bar{u}$ it is often possible to get a rate of convergence in (12) which is a straightforward consequence of the proof of Theorem 2.1.

Corollary 2.1. *Let the assumptions of Theorem 2.1 hold and assume that* $\bar{u}$ *satisfies conditions ensuring that in place of (8),*

$$\|(D_\alpha - a_\alpha r_\alpha)\, \bar{u}\,\|_\alpha = \omega(\alpha) c(\alpha) \to 0 \quad \textit{as} \ \ \alpha \to 0^+, \tag{16}$$

for $\omega(\alpha) > 0$ *defined for* α *sufficiently small and* $\omega(\alpha) \to 0$ *as* $\alpha \to 0^+$. *Then*

$$\|u_\alpha - r_\alpha \bar{u}\|_\alpha = \mathcal{O}(\omega(\alpha)) \quad \textit{as} \ \ \alpha \to 0^+,$$

where u_α denotes the solution of equation (10) with f^δ replaced by f. Further, for $\alpha = \alpha(\delta)$ selected so that (11) holds, it follows that

$$\left\| u^\delta_{\alpha(\delta)} - r_{\alpha(\delta)} \bar{u} \right\|_{\alpha(\delta)} = \mathcal{O}\left(\frac{\delta + \hat{\omega}(\alpha(\delta))}{c(\alpha(\delta))} \right) \to 0 \quad \text{as } \delta \to 0, \tag{17}$$

where $\hat{\omega}(\alpha) = \omega(\alpha) c(\alpha)$ and u^δ_α is the solution of (10).

Ideally such an $\alpha(\delta)$ can be found which balances the two terms in the numerator in (17).

3. The generalized method applied to the ν-smoothing Volterra problem

We now establish convergence of a 0^{th} order local regularization method for solving the ν-smoothing problem in $L^p(0,1), 1 \le p \le \infty$. Rather than develop a convergence theory directly, we instead show that this method of local regularization satisfies assumptions **A1**–**A3** in Section 2. We then apply Theorem 2.1 to establish convergence and use Corollary 2.1 to obtain convergence rates.

3.1. Selection of $[X_\alpha, r_\alpha]$, and T_α.

Let $\mathcal{A}$ be defined as in (3) for some fixed $\nu \ge 1$, and let $X = L^p(0,1)$, for fixed $1 \le p \le \infty$, with the usual norm denoted by $\|\cdot\|$. In the case of the Volterra problem considered here, it is natural for the reconstruction of $\bar{u}$ near $t \in (0,1)$ to make use of the data $f^\delta \in X$ restricted to the interval $(t, t+\alpha) \subseteq (0,1)$ for some value of $\alpha > 0$. We return to (6) as a starting point and take $T_\alpha f^\delta(t)$ to be the generalized average of f^δ over $(t, t+\alpha)$ for a.e. $t \in (0, 1-\alpha)$.

In particular, let $\bar{\alpha} > 0$ be small. Then for arbitrary $\alpha \in (0, \bar{\alpha}]$ and any $g \in X$, we define

$$T_\alpha g(t) := \frac{1}{\gamma_\alpha} \int_0^\alpha g(t+\rho)\, d\eta_\alpha(\rho), \qquad \text{a.e. } t \in (0, 1-\alpha), \tag{18}$$

where

$$\gamma_\alpha := \int_0^\alpha d\eta_\alpha(\rho),$$

and η_α is a signed measure specified below so that $\gamma_\alpha \ne 0$ and **A2** are satisfied. Then $T_\alpha \in \mathcal{L}(X, X_\alpha)$ where

$$X_\alpha := L^p(0, 1-\alpha) \tag{19}$$

with the usual norm $\|\cdot\|_\alpha$. A natural choice of $r_\alpha \in \mathcal{L}(X, X_\alpha)$ satisfying condition **A1** is given by the restriction operator

$$r_\alpha u(t) := u(t), \qquad \text{a.e. } t \in (0, 1-\alpha), \tag{20}$$

for $u \in X$.

3.2. Selection of $D_\alpha, \mathcal{A}_\alpha$, and a_α

For a.e. $t \in (0, 1-\alpha)$,

$$\begin{aligned} T_\alpha \mathcal{A}u(t) &= \frac{1}{\gamma_\alpha}\int_0^\alpha \int_0^{t+\rho} k(t+\rho-s)u(s)\,ds\,d\eta_\alpha(\rho) \\ &= \frac{1}{\gamma_\alpha}\int_0^\alpha \int_t^{t+\rho} k(t+\rho-s)u(s)\,ds\,d\eta_\alpha(\rho) \\ &\quad + \frac{1}{\gamma_\alpha}\int_0^t \int_0^\alpha k(t+\rho-s)\,d\eta_\alpha(\rho)\,u(s)\,ds, \end{aligned}$$

where the order of integration is changed in the second term above. Thus,

$$T_\alpha \mathcal{A} = D_\alpha + \mathcal{A}_\alpha r_\alpha,$$

where for $u \in X$ we define for a.e. $t \in (0, 1-\alpha)$,

$$D_\alpha u(t) := \frac{1}{\gamma_\alpha}\int_0^\alpha \int_0^\rho k(\rho - s)u(t+s)\,ds\,d\eta_\alpha(\rho), \tag{21}$$

and for $u \in X_\alpha$,

$$\mathcal{A}_\alpha u(t) := \int_0^t k_\alpha(t-s)u(s)\,ds, \qquad \text{a.e. } t \in (0, 1-\alpha), \tag{22}$$

where the kernel k_α for $\mathcal{A}_\alpha$ is given by

$$k_\alpha(t) := T_\alpha k(t) = \frac{1}{\gamma_\alpha}\int_0^\alpha k(t+\rho)\,d\eta_\alpha(\rho), \quad t \in (0, 1-\alpha). \tag{23}$$

Note that $D_\alpha u(t)$ only makes use of u on the local interval $(t, t+\alpha)$, and for $\alpha > 0$ small, one expects that the approximation

$$D_\alpha \bar{u}(t) \approx a_\alpha \bar{u}(t), \qquad \text{a.e. } t \in (0, 1-\alpha),$$

holds, where

$$a_\alpha := \frac{1}{\gamma_\alpha}\int_0^\alpha \int_0^\rho k(\rho - s)\,ds\,d\eta_\alpha(\rho). \tag{24}$$

Clearly, $k_\alpha \in C^\nu[0, 1-\alpha]$, and the operators $\mathcal{A}_\alpha$ and D_α are bounded linear, with $\mathcal{A}_\alpha \in \mathcal{L}(X_\alpha)$ and $D_\alpha \in \mathcal{L}(X, X_\alpha)$.

3.3. Local-regularizing families of measures and verification of Assumption **A2**

It remains to select a family of measures $\{\eta_\alpha\}_{\alpha\in(0,\bar{\alpha}]}$ for which $\gamma_\alpha \neq 0$ and assumptions **A2** and **A3** from Section 2 hold for all $\alpha \in (0, \bar{\alpha}]$. We define the notion of a local-regularizing family of measures for equation (10).

Definition 3.1. *A collection of signed measures* $\{\eta_\alpha\}_{\alpha\in(0,\bar{\alpha}]}$ *is said to be a* ***local-regularizing family of measures*** *for equation* (10) *if it satisfies the following properties:*

(i) There exists a $\sigma \in \mathbb{R}$ such that for each $j = 0, 1, ..., \nu$,

$$\int_0^\alpha \rho^j d\eta_\alpha(\rho) = \alpha^{j+\sigma} c_j \left(1 + C_j(\alpha)\right) \quad \text{for all} \quad \alpha \in (0, \bar{\alpha}],$$

where

(a) $C_j(\alpha)$ is a function for which there is a constant $\bar{C}_j \geq 0$ so that

$$|C_j(\alpha)| \leq \bar{C}_j \alpha < 1 \quad \text{for all} \quad \alpha \in (0, \bar{\alpha}];$$

(b) the constants $c_0, c_1, \ldots, c_\nu \in \mathbb{R}$ and $c_\nu \neq 0$ are such that the roots of the polynomial $p_\nu(\lambda)$, defined by

$$p_\nu(\lambda) = \frac{c_\nu}{\nu!}\lambda^\nu + \frac{c_{\nu-1}}{(\nu-1)!}\lambda^{\nu-1} + ... + \frac{c_1}{1!}\lambda + \frac{c_0}{0!}, \tag{25}$$

have negative real part.

(ii) There exists a constant $\tilde{C} > 0$ such that for each $\alpha \in (0, \bar{\alpha}]$,

$$|\eta_\alpha|(0, \alpha) = \int_0^\alpha d|\eta_\alpha|(\rho) \leq \tilde{C}\alpha^\sigma,$$

where $|\eta_\alpha|$ denotes the total variation measure.

Remark 3.1. *The above definition is a slight modification of hypotheses (H1)-(H3) given in [15].*

Henceforth, we assume without loss of generality that $c_\nu = \nu!$ in Definition 3.1.

Proposition 3.1. *If $\{\eta_\alpha\}_{\alpha \in (0,\bar{\alpha}]}$ is a local-regularizing family of measures for* (10)*, then $\gamma_\alpha > 0$ for each $\alpha \in (0, \bar{\alpha}]$. Moreover, the family $\{T_\alpha\}_{\alpha \in (0,\bar{\alpha}]}$ defined by* (18) *satisfies* **A2** *with*

$$M_T = \frac{\tilde{C}}{c_0\left(1 - \bar{C}_0\bar{\alpha}\right)}. \tag{26}$$

Proof. Let $-m_1, \ldots, -m_\nu$, be the roots of the polynomial p_ν in (25) so that $\Re(m_i) > 0$, for $i = 1, \ldots, \nu$. Then $p_\nu(\lambda) = \prod_{i=1}^{\nu}(\lambda + m_i)$ since $c_\nu = \nu!$ and

$$c_0 = \prod_{i=1}^{\nu} m_i > 0.$$

It follows directly from part (i) in Definition 3.1 that

$$\gamma_\alpha \geq \alpha^\sigma c_0(1 - \bar{C}_0\bar{\alpha}) > 0$$

for each $\alpha \in (0, \bar{\alpha}]$, proving the first assertion.

Let $g \in X$. By Minkowski's integral inequality and part (ii) of Definition 3.1,

$$\begin{aligned}
\|T_\alpha g\|_\alpha &\leq \frac{1}{\gamma_\alpha}\left\|\int_0^\alpha |g(\cdot+\rho)| d|\eta_\alpha|(\rho)\right\|_\alpha \\
&\leq \frac{1}{\gamma_\alpha}\int_0^\alpha \|g(\cdot+\rho)\|_\alpha \, d\left|\eta_\alpha\right|(\rho) \\
&\leq \frac{1}{\gamma_\alpha}\|g\| \int_0^\alpha d|\eta_\alpha|(\rho) \\
&\leq \frac{\tilde{C}}{c_0\left(1 - \bar{C}_0\bar{\alpha}\right)}\|g\|.
\end{aligned}$$

□

Classes of measures satisfying Definition 3.1 can always be chosen as follows.

Proposition 3.2. *Let $\nu \in \mathbb{N}$. Suppose the constants $c_0, c_1, \ldots, c_\nu \in \mathbb{R}$ and $c_\nu > 0$ are such that the roots of the polynomial $p_\nu(\lambda)$ in (25) have negative real part. Then there exists $\psi \in L^1(0,1)$ such that the collection of measures $\{\eta_\alpha\}_{\alpha\in(0,\bar{\alpha}]}$ defined by*

$$d\eta_\alpha(\rho) = \psi\left(\frac{\rho}{\alpha}\right) d\rho, \ \text{ a.e. } \rho \in (0,\alpha), \ \alpha \in (0,\bar{\alpha}], \tag{27}$$

is a local-regularizing family of measures for (10), where $\sigma = 1$, $C_i(\alpha) = \mathbf{0}$ for all $i = 0, 1, \ldots, \nu$, and $\tilde{C} = \|\psi\|_{L^1(0,1)}$.

Proof. Existence of a νth degree polynomial $\tilde{\psi}$ on $[0,1]$ such that

$$\int_0^1 \rho^j \tilde{\psi}(\rho)\, d\rho = c_j \ \text{ for } \ j = 0, \ldots, \nu, \tag{28}$$

follows from the arguments in Lemma 2.2 of [15]. Let $\psi \in L^1(0,1)$ be any such function for which (28) holds, and for each $\alpha \in (0,\bar{\alpha}]$, define η_α by (27). Then

$$\begin{aligned}\int_0^\alpha \rho^j d\eta_\alpha(\rho) &= \int_0^\alpha \rho^j \psi\left(\frac{\rho}{\alpha}\right) d\rho \\ &= \int_0^1 (\alpha\rho)^j \psi(\rho)\, \alpha d\rho \\ &= \alpha^{j+1} c_j\end{aligned}$$

for $j = 0, 1, ..., \nu$. A change of variables yields

$$|\eta_\alpha|(0,\alpha) = \int_0^\alpha \left|\psi\left(\frac{\rho}{\alpha}\right)\right| \, d\rho = \alpha \|\psi\|_{L^1(0,1)}\,.$$

□

The standard construction of local-regularizing measures involves finding a polynomial ψ of degree ν with coefficients satisfying the particular matrix equation arising from (28) (see [15]). However, the leading matrix in this equation is an ill-conditioned Hilbert matrix, which for ν large can lead to erratic behavior of the constructed ψ, especially near the origin. We address this issue in the next lemma by providing a more stable method with which to generate ψ. In the following we let $P(\cdot;\mathbf{d})$ denote the polynomial

$$P(\lambda;\mathbf{d}) := d_\nu\lambda^\nu + d_{\nu-1}\lambda^{\nu-1} + \cdots + d_1\lambda + d_0 \tag{29}$$

for some $\mathbf{d} = (d_0, d_1, \ldots, d_\nu)^\top \in \mathbb{R}^{\nu+1}$.

Lemma 3.1. *For arbitrary negative constants $-m_1, -m_2, \ldots, -m_\nu$, let $\bar{\mathbf{d}} = (\bar{d}_0, \bar{d}_1, \ldots, \bar{d}_\nu)^\top \in \mathbb{R}^{\nu+1}$, $\bar{d}_\nu = 1$, be such that*

$$P(\lambda;\bar{\mathbf{d}}) = \prod_{i=1}^{\nu} (\lambda + m_i)\,,$$

for the polynomial P defined in (29). Further, let $\bar{\mathbf{a}} = (\bar{a}_0, \bar{a}_1, \dots, \bar{a}_\nu)^\top \in \mathbb{R}^{\nu+1}$ be the unique solution of the matrix equation

$$\mathbf{H}\mathbf{a} = \bar{\mathbf{d}},$$

where $\mathbf{H}$ is the nonsingular $(\nu+1)$-square Hilbert matrix with entries $\mathbf{H}_{i,j} = 1/(i+j+1)$. Then for each $\beta > 0$, there is a unique solution $\mathbf{a}(\beta) = (a_0(\beta), a_1(\beta), \dots, a_\nu(\beta))^\top \in \mathbb{R}^{\nu+1}$ of the Tikhonov problem

$$\min_{\mathbf{a}\in\mathbb{R}^{\nu+1}} \left\{ \|\mathbf{H}\mathbf{a} - \bar{\mathbf{d}}\|^2_{\mathbb{R}^{\nu+1}} + \beta\|\mathbf{a}\|^2_{\mathbb{R}^{\nu+1}} \right\}, \tag{30}$$

satisfying $\|\mathbf{a}(\beta)\|_{\mathbb{R}^{\nu+1}} \le \|\bar{\mathbf{a}}\|_{\mathbb{R}^{\nu+1}}$. Moreover, for $\beta > 0$ sufficiently small, the polynomial $\psi(\cdot;\beta)$ defined by

$$\psi(\rho;\beta) = \sum_{i=0}^{\nu} a_i(\beta)\rho^i,$$

generates a (β-dependent) family $\{\eta_\alpha\}$ of local-regularizing measures defined by

$$d\eta_\alpha(\rho) = \psi(\rho/\alpha;\beta)\, d\rho, \tag{31}$$

with $c_\nu > 0$.

Proof. By construction, all roots of the polynomial $P(\cdot;\bar{\mathbf{d}})$ are negative. It follows from a continuity argument that there exists $\varepsilon > 0$ so that for any $\mathbf{d} = (d_0, d_1, \dots, d_\nu)^\top \in \mathbb{R}^{\nu+1}$ satisfying

$$\|\bar{\mathbf{d}} - \mathbf{d}\|_{\mathbb{R}^{\nu+1}} < \varepsilon, \tag{32}$$

all roots of the polynomial $P(\cdot;\mathbf{d})$ have negative real part. If needed, we decrease the value of $\varepsilon > 0$ even further so that $d_\nu > 0$ for all such $\mathbf{d}$.

For $\beta > 0$, let $\mathbf{a}(\beta) \in \mathbb{R}^{\nu+1}$ be the unique solution of the Tikhonov problem (30). Then

$$\begin{aligned}\max\left\{\|\mathbf{H}\mathbf{a}(\beta) - \bar{\mathbf{d}}\|^2_{\mathbb{R}^{\nu+1}},\ \beta\|\mathbf{a}(\beta)\|^2_{\mathbb{R}^{\nu+1}}\right\} &\\ \le \|\mathbf{H}\mathbf{a}(\beta) - \bar{\mathbf{d}}\|^2_{\mathbb{R}^{\nu+1}} &+ \beta\|\mathbf{a}(\beta)\|^2_{\mathbb{R}^{\nu+1}} \\ \le \|\mathbf{H}\bar{\mathbf{a}} - \bar{\mathbf{d}}\|^2_{\mathbb{R}^{\nu+1}} &+ \beta\|\bar{\mathbf{a}}\|^2_{\mathbb{R}^{\nu+1}} \\ = \beta\|\bar{\mathbf{a}}\|^2_{\mathbb{R}^{\nu+1}}&.\end{aligned}$$

For $\beta \in \left(0, [\varepsilon/\|\bar{\mathbf{a}}\|_{\mathbb{R}^{\nu+1}}]^2\right)$, define $\mathbf{d}(\beta) := \mathbf{H}\mathbf{a}(\beta)$. Then

$$\|\mathbf{d}(\beta) - \bar{\mathbf{d}}\|_{\mathbb{R}^{\nu+1}} \le \beta^{1/2}\|\bar{\mathbf{a}}\|_{\mathbb{R}^{\nu+1}} < \varepsilon,$$

guaranteeing that all roots of the the polynomial $P(\cdot, \mathbf{d}(\beta))$ have negative real part and $d_\nu > 0$. Then since $\mathbf{a}(\beta)$ solves $\mathbf{H}\mathbf{a} = \mathbf{d}(\beta)$ and $\mathbf{a}(\beta) \ne \mathbf{0}$, it follows that the family $\{\eta_\alpha\}$ of measures defined by (31) is local-regularizing with $c_j = d_j j!$ for all $j = 0, \dots, \nu$ and $c_\nu > 0$. □

Remark 3.2. *From $\psi(0;\beta) = a_0(\beta)$, we have $|\psi(0;\beta)| \le \|\bar{\mathbf{a}}\|_{\mathbb{R}^{\nu+1}}$, imposing some control on the polynomial near the origin.*

3.4. Verification of Assumption **A3**(i)

Henceforth, let $\{\eta_\alpha\}_{\alpha\in(0,\bar{\alpha}]}$ be a local-regularizing family of measures for equation (10). For arbitrary $\alpha \in (0,\bar{\alpha}]$, let T_α, X_α, r_α, $\mathcal{A}_\alpha$, and a_α be as defined in (18)-(20), (22), and (24), respectively. We begin the verification of assumption **A3** by showing that $a_\alpha > 0$. The form of this proof, which is needed later, differs from that found in [14, 15, 22].

Lemma 3.2. *There exists a constant* $\bar{\kappa} > 0$ *for which*

$$\|k\|_{C[0,\alpha]} \leq \bar{\kappa}\alpha^{\nu-1} \tag{33}$$

for every $\alpha \in (0,\bar{\alpha}]$*. Further, if* $\bar{\alpha} > 0$ *is sufficiently small, then there exist constants* $0 < \kappa_1 < \kappa_2$ *such that*

$$\kappa_1\alpha^\nu \leq a_\alpha \leq \kappa_2\alpha^\nu,$$

for all $\alpha \in (0,\bar{\alpha}]$*.*

Proof. The result in (33) with $\bar{\kappa} = \left(\nu + \bar{\alpha}\left\|k^{(\nu)}\right\|_{C[0,\bar{\alpha}]}\right)/\nu!$ follows trivially from the Taylor expansion about 0 of the general ν-smoothing kernel $k \in C^\nu[0,1]$,

$$k(t) = \frac{t^{\nu-1}}{(\nu-1)!} + k^{(\nu)}(\zeta_t)\frac{t^\nu}{\nu!},$$

for some $\zeta_t \in (0,t)$ and each $t \in [0,1]$.

By Definition 3.1, we obtain for any $\alpha \in (0,\bar{\alpha}]$,

$$\begin{aligned} a_\alpha &= \frac{1}{\gamma_\alpha}\left[\int_0^\alpha \frac{\rho^\nu}{\nu!}\,d\eta_\alpha(\rho) + \int_0^\alpha \left(\int_0^\rho k^{(\nu)}(\zeta_s)\frac{s^\nu}{\nu!}\,ds\right)d\eta_\alpha(\rho)\right] \\ &= \frac{1}{\gamma_\alpha}\left[\alpha^{\nu+\sigma}(1 + C_\nu(\alpha)) + \int_0^\alpha \left(\int_0^\rho k^{(\nu)}(\zeta_s)\frac{s^\nu}{\nu!}ds\right)d\eta_\alpha(\rho)\right], \end{aligned} \tag{34}$$

Then from the bounds on γ_α,

$$a_\alpha \leq \frac{\alpha^\nu}{c_0(1-\bar{C}_0\bar{\alpha})}\left[1 + \bar{C}_\nu\bar{\alpha} + \left\|k^{(\nu)}\right\|_{C[0,\bar{\alpha}]}\frac{\tilde{C}\bar{\alpha}}{(\nu+1)!}\right],$$

$$a_\alpha \geq \frac{\alpha^\nu}{c_0(1+\bar{C}_0\bar{\alpha})}\left[1 - \bar{C}_\nu\bar{\alpha} - \left\|k^{(\nu)}\right\|_{C[0,\bar{\alpha}]}\frac{\tilde{C}\bar{\alpha}}{(\nu+1)!}\right],$$

so that a_α is strictly positive for $\bar{\alpha} > 0$ sufficiently small. □

The next corollary establishes that assumption **A3**(i) holds, a result which follows directly from the fact that

$$a_\alpha u + \mathcal{A}_\alpha u = h, \tag{35}$$

is a well-posed equation for arbitrary $h \in X_\alpha$ (e.g., [9], p. 44).

Corollary 3.1. *Let* $\bar{\alpha} > 0$ *be sufficiently small so that* $a_\alpha \neq 0$ *for all* $\alpha \in (0,\bar{\alpha}]$*. Then the operator* $(a_\alpha + \mathcal{A}_\alpha) \in \mathcal{L}(X_\alpha)$ *has a bounded inverse, with*

$$\left\|(a_\alpha + \mathcal{A}_\alpha)^{-1}\right\|_{\mathcal{L}(X_\alpha)} \leq \frac{1}{c(\alpha)}, \tag{36}$$

for some $c(\alpha) > 0$*.*

3.5. Verification of Assumption **A3**(ii)

We now show that for $1 < p < \infty$, or with increased regularity of $\bar{u}$ *for all* $1 \le p \le \infty$, the local regularization method introduced here satisfies assumption **A3**(ii).

In the following lemma we obtain bounds which are used to estimate $c(\alpha)$ in Corollary 3.2 and also to derive convergence rates under general source conditions in Theorem 3.2. We note that while Corollary 3.2 can also be established using a proof similar to that of Theorem 3.1 in [15] (which extends the arguments establishing Lemma 1 in [22]), a new proof involving Laplace transforms is needed here in order to jointly obtain the results in Corollary 3.2 and Theorem 3.2 below.

Lemma 3.3. *There exists $\bar{\alpha} > 0$ sufficiently small such that for all $\alpha \in (0, \bar{\alpha}]$, if $h \in C^m[0, 1-\alpha]$ for some integer $m = 1, \ldots, \nu$, there exists a unique solution $y_\alpha \in C[0, 1-\alpha]$ of*

$$y(t) + \int_0^t \frac{k_\alpha(t-s)}{a_\alpha} y(s)\, ds = \frac{h(t)}{a_\alpha}, \quad t \in (0, 1-\alpha], \tag{37}$$

where k_α is defined in (23). Moreover, y_α satisfies

$$\|y_\alpha\|_\alpha \le \hat{C}_p\, M_p(\alpha; h), \tag{38}$$

where, in the case of $p = \infty$,

$$M_\infty(\alpha; h) = \max\left\{ \frac{\left\|h^{(m)}\right\|_{L^\infty(0,1-\alpha)}}{\alpha^{\nu-m}},\ \frac{\left|h^{(j)}(0)\right|}{\alpha^{\nu-j}},\ j = 0, \ldots, m-1 \right\},$$

whereas for $1 \le p < \infty$,

$$M_p(\alpha; h) = \alpha^{1/p} \max\left\{ \frac{\left\|h^{(m)}\right\|_{L^\infty(0,1-\alpha)}}{\alpha^{\nu-m+1}},\ \frac{\left|h^{(j)}(0)\right|}{\alpha^{\nu-j}},\ j = 0, \ldots, m-1 \right\},$$

for $\hat{C}_p$ independent of α, but dependent on k, ν, and the family $\{\eta_\alpha\}_{\alpha \in (0,\bar{\alpha}]}$.

Proof. See Appendix. □

We now apply the results of Lemma 3.3 to estimate $c(\alpha)$.

Corollary 3.2. *For $\bar{\alpha} > 0$ sufficiently small, the bound on $\|(a_\alpha + \mathcal{A}_\alpha)^{-1}\|_{\mathcal{L}(X_\alpha)}$ in (36) holds for all $\alpha \in (0, \bar{\alpha}]$, with*

$$c(\alpha) = C a_\alpha$$

for some $C > 0$ independent of α, but dependent on k, ν, and the family $\{\eta_\alpha\}_{\alpha \in (0,\bar{\alpha}]}$.

Proof. Assume that Lemma 3.2 holds and let $\alpha \in (0, \bar{\alpha}]$. Since $k_\alpha \in C^\nu[0, 1-\alpha]$, it follows from Lemma 3.3 that there exists a unique (continuous) function $\mathcal{X}_\alpha \in L^1(0, 1-\alpha)$, called the *resolvent kernel* satisfying the (resolvent) equation

$$\mathcal{X}_\alpha(t) + \int_0^t \frac{k_\alpha(t-s)}{a_\alpha} \mathcal{X}_\alpha(s) ds = \frac{k_\alpha(t)}{a_\alpha}, \quad t \in (0, 1-\alpha), \tag{39}$$

for $\bar{\alpha} > 0$ sufficiently small. (See [7, 9] for properties of resolvent kernels.) In addition, for given $h \in X_\alpha$ the unique solution $u(\cdot;\alpha,h) \in X_\alpha$ of (35),

$$u(\cdot;\alpha,h) = (a_\alpha I + A_\alpha)^{-1} h,$$

may be written using the *variation of constants formula* (e.g., [6, 9]) as

$$u(\cdot;\alpha,h) = \frac{h}{a_\alpha} - \mathcal{X}_\alpha \star \frac{h}{a_\alpha}.$$

It then follows that

$$\left\| (a_\alpha + \mathcal{A}_\alpha)^{-1} \right\|_{\mathcal{L}(X_\alpha)} \le \frac{1}{a_\alpha} \left(1 + \|\mathcal{X}_\alpha\|_{L^1(0,1-\alpha)} \right),$$

where it remains to estimate $\|\mathcal{X}_\alpha\|_{L^1(0,1-\alpha)}$ from Lemma 3.3.

We make use of the bound in (38) with $p = 1$, $h = k_\alpha$, and $m = \nu$, as well as (64) and (67), and note that for $j = 0, 1, \ldots, \nu - 1$,

$$\begin{aligned} \frac{\alpha^{1/p}}{\alpha^{\nu-j}} \left| k_\alpha^{(j)}(0) \right| &= \frac{a_\alpha}{\alpha^\nu} \cdot \frac{\alpha^{j+1}}{a_\alpha} \left| k_\alpha^{(j)}(0) \right| \\ &\le \kappa_2 \left(\frac{c_{\nu-j-1}}{(\nu-j-1)!} + \xi_{\nu-j-1} \bar{\alpha} \right). \end{aligned}$$

In addition, $\left\| k_\alpha^{(\nu)} \right\|_{L^\infty(0,1-\alpha)}$ is uniformly bounded in α as shown in (59). Thus the resolvent kernel $\mathcal{X}_\alpha$ defined in equation (39) satisfies

$$\|\mathcal{X}_\alpha\|_{L^1(0,1-\alpha)} \le M,$$

for all $\alpha \in (0, \bar{\alpha}]$ and some $M > 0$ independent of α, so that the inequality in (36) holds for $c(\alpha) = C a_\alpha$, where $C = 1/(1+M)$. □

We complete verification of **A3**(ii) with the following lemmas. In the first lemma, we provide an argument involving the Hardy-Littlewood Maximal theorem, which only applies appropriately to our problem in the case $1 < p < \infty$. We include the cases $p = 1, \infty$ in a second lemma. Henceforth, we assume that $\bar{\alpha}$ is sufficiently small so that Lemma 3.2 and Corollary 3.2 hold.

Lemma 3.4. *Let $c(\alpha) = C a_\alpha$ for all $\alpha \in (0, \bar{\alpha}]$. If $1 < p < \infty$, then*

$$\left\| (D_\alpha - a_\alpha r_\alpha)\, \bar{u} \right\|_\alpha = o\left(c(\alpha)\right) \quad \text{as } \alpha \to 0^+. \tag{40}$$

Proof. For arbitrary $\alpha \in (0, \bar{\alpha}]$ and for a.e. $t \in (0, 1-\alpha)$,

$$\begin{aligned} & D_\alpha \bar{u}(t) - a_\alpha r_\alpha \bar{u}(t) \\ & \le \frac{1}{\gamma_\alpha} \int_0^\alpha \left| \int_0^\rho k(\rho - s) \left(\bar{u}(t+s) - \bar{u}(t) \right) \, ds \right| \, d\left|\eta_\alpha\right|(\rho) \\ & \le \frac{1}{\gamma_\alpha} \sup_{\rho \in (0,\alpha)} \left| \int_0^\rho k(\rho - s) \left(\bar{u}(t+s) - \bar{u}(t) \right) \, ds \right| \tilde{C} \alpha^\sigma \\ & \le \frac{\tilde{C}}{c_0 \left(1 - \bar{C}_0 \bar{\alpha} \right)} \|k\|_{C[0,\alpha]} \int_0^\alpha \left| \bar{u}(t+s) - \bar{u}(t) \right| ds \\ & \le M_T \bar{\kappa} \alpha^{\nu-1} \int_0^\alpha \left| \bar{u}(t+s) - \bar{u}(t) \right| ds, \end{aligned} \tag{41}$$

which follows from Definition 3.1, Lemma 3.2, and the definition of M_T in (26). Thus for $1 < p < \infty$,

$$\begin{aligned}\|D_\alpha \bar{u} - a_\alpha r_\alpha \bar{u}\|_\alpha &\leq \alpha^\nu M_T \bar{\kappa} \left[\int_0^{1-\alpha} \left(\frac{1}{\alpha}\int_0^\alpha |\bar{u}(t+s) - \bar{u}(t)|\ ds\right)^p dt\right]^{1/p} \\ &\leq \alpha^\nu M_T \bar{\kappa} \left[\int_0^{1} \left(\frac{1}{\alpha}\int_0^\alpha |\bar{u}_{\mathrm{ext}}(t+s) - \bar{u}_{\mathrm{ext}}(t)|\ ds\right)^p dt\right]^{1/p},\end{aligned}$$

where $\bar{u}_{\mathrm{ext}} \in L^p(0,\infty)$ is defined via $\bar{u}_{\mathrm{ext}} = \begin{cases} \bar{u}(t), & \text{a.e. } t \in (0,1), \\ 0, & \text{a.e. } t \in [1,\infty). \end{cases}$

But $\alpha^\nu = \mathcal{O}(a_\alpha)$, so if we demonstrate that

$$\lim_{\alpha\to 0} \left[\int_0^{1} \left(\frac{1}{\alpha}\int_0^\alpha |\bar{u}_{\mathrm{ext}}(t+s) - \bar{u}_{\mathrm{ext}}(t)|\, ds\right)^p dt\right]^{1/p} = 0, \tag{42}$$

the result in (40) is obtained.

By Lebesgue's differentiation theorem,

$$\lim_{\alpha\to 0} \frac{1}{\alpha}\int_0^\alpha |\bar{u}_{\mathrm{ext}}(t+s) - \bar{u}_{\mathrm{ext}}(t)|\, ds = 0, \qquad \text{a.e. } t \in (0,1),$$

so that

$$\lim_{\alpha\to 0} \varphi_\alpha(t) = 0, \qquad \text{a.e. } t \in (0,1),$$

with φ_α defined by

$$\varphi_\alpha(t) := \left(\frac{1}{\alpha}\int_0^\alpha |\bar{u}_{\mathrm{ext}}(t+s) - \bar{u}_{\mathrm{ext}}(t)|\, ds\right)^p, \qquad \text{a.e. } t \in (0,1).$$

Consider the function

$$\varphi(t) := \sup_{\alpha>0} \left(\frac{1}{\alpha}\int_0^\alpha |\bar{u}_{\mathrm{ext}}(t+s) - \bar{u}_{\mathrm{ext}}(t)|\, ds\right)^p, \qquad \text{a.e. } t \in (0,1),$$

and note that

$$\begin{aligned}\varphi(t) &\leq 2^p \left[\sup_{\alpha>0} \left(\frac{1}{\alpha}\int_0^\alpha |\bar{u}_{\mathrm{ext}}(t+s)|\, ds\right)^p + |\bar{u}_{\mathrm{ext}}(t)|^p\right] \\ &= 2^p \left[(M_+ \bar{u}_{\mathrm{ext}}(t))^p + |\bar{u}_{\mathrm{ext}}(t)|^p\right],\end{aligned}$$

for a.e. $t \in (0,1)$, where

$$M_+ \bar{u}_{\mathrm{ext}}(t) := \sup_{\alpha>0} \left(\frac{1}{\alpha}\int_t^{t+\alpha} |\bar{u}_{\mathrm{ext}}(s)|\, ds\right), \qquad \text{a.e. } t \in (0,\infty),$$

is the one-sided Hardy-Littlewood maximal function of $\bar{u}_{\mathrm{ext}}$. Then by the Hardy-Littlewood Maximal Theorem for $p \in (1,\infty)$ [10] and for some $C_p > 0$ independent of α, we have

$$\begin{aligned}\int_0^1 |\varphi(t)|\ dt &\leq 2^p \int_0^1 \left[(M_+ \bar{u}_{\mathrm{ext}}(t))^p + |\bar{u}_{\mathrm{ext}}(t)|^p\right]\ dt \\ &\leq 2^p \left(C_p + 1\right) \|\bar{u}\|^p_{L^p(0,1)} \\ &< \infty,\end{aligned}$$

since $\bar{u} \in L^p(0,1)$.

Hence $\varphi(t) \in L^1(0,1)$ and, for each α, $\varphi_\alpha(t) \leq \varphi(t)$ for a.e. $t \in (0,1)$. An application of the Lebesgue Dominated Convergence Theorem yields the desired convergence in (42) for $p \in (1,\infty)$. □

Condition **A3**(ii) also holds in the cases $p = 1$ and $p = \infty$ assuming continuity of $\bar{u}$ on $[0,1]$.

Lemma 3.5. *Let $c(\alpha) = Ca_\alpha$ for all $\alpha \in (0,\bar{\alpha}]$. If $\bar{u} \in C[0,1]$, then*

$$\|(D_\alpha - a_\alpha r_\alpha)\,\bar{u}\|_\alpha = o\,(c(\alpha)) \quad \text{as } \alpha \to 0^+, \tag{43}$$

for all $1 \leq p \leq \infty$.

Proof. Let $\epsilon > 0$ and define a modulus of continuity for $\bar{u}$,

$$\omega(\epsilon,\bar{u}) := \sup\left\{|\bar{u}(t) - \bar{u}(\tau)|\,;\ t,\tau \in (0,1),\ |t-\tau| \leq \epsilon\right\}.$$

From (41) and the assumption on $\bar{u}$, it follows that for every $\alpha \in (0,\bar{\alpha}]$ and all $t \in (0, 1-\alpha)$,

$$\begin{aligned} D_\alpha\bar{u}(t) - a_\alpha r_\alpha \bar{u}(t) &\leq M_T\bar{\kappa}\alpha^{\nu-1}\int_0^\alpha \omega(s,\bar{u})ds \\ &\leq M_T\bar{\kappa}\alpha^{\nu} \sup_{s\in[0,\alpha]} \omega(s,\bar{u}). \end{aligned} \tag{44}$$

Since $\alpha^\nu = O(a_\alpha)$ and $\lim_{\alpha\to 0}\sup_{s\in[0,\alpha]} \omega(s,\bar{u}) = 0$, the assertion holds for all $1 \leq p \leq \infty$. □

With the verification of **A1**–**A3**, convergence of the local regularization method for the ν-smoothing problem follows from Theorem 2.1.

Theorem 3.1. *Let $\{\eta_\alpha\}_{\alpha\in(0,\bar{\alpha}]}$ be a local-regularizing family of measures for equation (10), and let T_α, X_α, r_α, $\mathcal{A}_\alpha$, and a_α be as defined in (18)-(20), (22), and (24), respectively. Let $f^\delta \in X$ satisfy $\|f - f^\delta\| \leq \delta$ for some $\delta > 0$. Then for $\bar{\alpha} > 0$ sufficiently small and for $\alpha \in (0,\bar{\alpha}]$, there exists a unique solution u_α^δ of (10) for which*

$$\|u_\alpha^\delta - r_\alpha\bar{u}\|_\alpha \to 0 \quad \text{as } \delta \to 0,$$

provided $\alpha = \alpha(\delta)$ is selected satisfying (11) and where $\|\cdot\|_\alpha$ is the $L^p(0,1-\alpha)$ norm for $1 < p < \infty$. In addition, if $\bar{u} \in C[0,1]$, then the convergence is obtained for all $1 \leq p \leq \infty$.

3.6. Convergence rates

We now give conditions on $\bar{u}$ so that rates of convergence are found using Corollary 2.1. In doing so, we make use of the Riemann-Liouville fractional integral

$$D^{-\mu}w(t) := \frac{1}{\Gamma(\mu)}\int_0^t (t-s)^{\mu-1}w(s)\,ds, \quad t \in [0,1],\ \mu > 0,$$

for $w \in C[0,1]$ and Γ the usual Gamma function. Note that if $v = D^{-\mu}w$ for $\mu > 0$, then $v \in C[0,1]$ as it is the convolution of $w \in C[0,1]$ and the function $t^{\mu} \in L^1(0,1)$ (see, e.g., Chapter 2 of [9]). Further, if

$$v = D^{-n}w,$$

for some $n = 1, 2, \ldots$, then $v \in C^n[0,1]$ with $v(0) = v'(0) = \cdots = v^{n-1}(0) = 0$, and $v^{(n)}(t) = w(t)$, $t \in [0,1]$.

Theorem 3.2. *Suppose that $\bar{u}$ satisfies one of the following:*

(i) $\bar{u}$ is Hölder continuous on $[0,1]$ with exponent $\mu \in (0,1]$ and constant $L_{\bar{u}}$; or

(ii) for some $w \in C[0,1]$ and $\mu \in (0, \nu+1]$,

$$\bar{u} = D^{-\mu}w. \tag{45}$$

Then

$$\|u_\alpha - r_\alpha \bar{u}\|_\alpha = \mathcal{O}\left(\alpha^{\mu}\right) \quad \text{as} \;\; \alpha \to 0, \tag{46}$$

where u_α solves the local regularization equation (10) with f^δ replaced by f. In addition, given $f^\delta \in X$ satisfying $\|f - f^\delta\| \le \delta$ for some $\delta > 0$, and for u_α^δ solving (10), the choice of $\alpha = \alpha(\delta) = K\delta^{1/(\mu+\nu)}$ for some constant $K > 0$ yields a rate of convergence

$$\left\|u_\alpha^\delta - r_\alpha \bar{u}\right\|_\alpha = \mathcal{O}\left(\delta^{\mu/(\mu+\nu)}\right) \quad \text{as} \;\; \delta \to 0. \tag{47}$$

In this case the optimal rate for the source condition (45) is obtained when $\mu = \nu + 1$.

Proof. Fix $\alpha \in (0, \bar{\alpha}]$.

(i) The first result follows from Lemma 3.5 with modulus of continuity in (44) given by $\omega(s, \bar{u}) = L_{\bar{u}} s^{\mu}$. Making this substitution into (41),

$$\begin{aligned}\|D_\alpha \bar{u} - a_\alpha r_\alpha \bar{u}\|_\alpha &\le \alpha^{\nu-1} M_T \bar{\kappa} \int_0^\alpha L_{\bar{u}} s^{\mu}\, ds \\ &= M_T \bar{\kappa} \frac{L_{\bar{u}} \alpha^{\nu+\mu}}{\mu+1} \\ &= \mathcal{O}\left(\alpha^{\nu+\mu}\right) \qquad \text{as} \;\; \alpha \to 0.\end{aligned}$$

Therefore

$$\begin{aligned}\|u_\alpha - r_\alpha \bar{u}\|_\alpha &\le \frac{1}{c(\alpha)} \|(D_\alpha - r_\alpha a_\alpha)\bar{u}\|_\alpha \\ &= \mathcal{O}\left(\alpha^{\mu}\right) \qquad \text{as} \;\; \alpha \to 0,\end{aligned}$$

using the fact that $c(\alpha) = \mathcal{O}\left(\alpha^{\nu}\right)$.

(ii) The rate in (46) for $\mu \in (0,1]$ is a consequence of the first part of the theorem once Hölder continuity of $\bar{u}$ is established. To that end, let $\mu \in (0,1]$. Then for all $t \in [0, 1-\alpha]$ and $\tau \in [0, \alpha]$,

$$\begin{aligned}&|\bar{u}(t) - \bar{u}(t+\tau)| \\ &= \frac{1}{\Gamma(\mu)} \left| \int_0^t \left[(t-s)^{\mu-1} - (t+\tau-s)^{\mu-1}\right] w(s)\, ds\right.\end{aligned}$$

$$
\begin{aligned}
&\qquad\qquad\qquad - \int_0^\tau (\tau - s)^{\mu-1} w(t+s)\, ds \Big| \\
&\leq \frac{\|w\|_{L^\infty(0,1)}}{\Gamma(\mu)} \left[\int_0^t s^{\mu-1} - (s+\tau)^{\mu-1}\, ds + \int_0^\tau (\tau - s)^{\mu-1}\, ds \right] \qquad (48) \\
&\leq \frac{\|w\|_{L^\infty(0,1)}}{\Gamma(\mu+1)} \left(t^\mu - (t+\tau)^\mu + 2\tau^\mu \right) \\
&\leq \frac{2\|w\|_{L^\infty(0,1)}}{\Gamma(\mu+1)} \tau^\mu.
\end{aligned}
$$

Thus $\bar{u}$ is Hölder continuous on the interval $[0,1]$ with exponent μ and constant $2\|w\|_{L^\infty(0,1)}/\Gamma(\mu+1)$. (Note that the constant 2 in the bound above can be improved by a factor of one-half in the case $\mu = 1$.)

We complete the proof for $\mu \in (1, \nu+1]$ using Corollary 2.1. To do so, we show that (16) holds with $\omega(\alpha) = \mathcal{O}(\alpha^\mu)$, i.e., that

$$
\|v_\alpha\|_\alpha = \mathcal{O}(\alpha^\mu) \quad \text{as } \alpha \to 0,
$$

where $v_\alpha = u_\alpha - r_\alpha \bar{u}$ is the solution in X_α of equation (35) using

$$
h(t;\alpha) := \frac{1}{\gamma_\alpha} \int_0^\alpha \int_0^\rho k(\rho - s) \left[\bar{u}(t+s) - \bar{u}(t) \right] ds\, d\eta_\alpha(\rho), \tag{49}
$$

for a.e. $t \in (0, 1-\alpha)$ and all $\alpha \in (0, \bar{\alpha}]$. With $u_\alpha \in L^\infty(0, 1-\alpha)$ and $r_\alpha \bar{u} \in C[0, 1-\alpha]$, the rate of convergence is established for $\|v_\alpha\|_{L^\infty(0,1-\alpha)}$ as $\alpha \to 0$ using Lemma 3.3. Some preliminary definitions and estimates are needed. Let $\mu \in (1, \nu+1]$ and define $m = m(\mu)$ via

$$
m := \begin{cases} \lfloor \mu \rfloor, & 1 < \mu < \nu + 1, \\ \nu, & \mu = \nu + 1. \end{cases}
$$

If $\mu = 2, 3, \ldots, \nu$, then $m = \mu$ and

$$
\bar{u}^{(m)}(t+\tau) - \bar{u}^{(m)}(t) = w(t+\tau) - w(t), \quad t \in [0, 1-\alpha],
$$

for all $\tau \in [0, \alpha]$ so that $h^{(m)} \in C[0, 1-\alpha]$.

For the remaining values of $\mu \in (1, \nu+1]$, $\mu \neq 2, 3, \ldots, \nu$, we have $\tilde{\mu} := \mu - m \in (0, 1]$ so that for all $t \in [0, 1-\alpha]$ and $\tau \in [0, \alpha]$,

$$
\begin{aligned}
&\bar{u}^{(m)}(t+\tau) - \bar{u}^{(m)}(t) \\
&\quad = \frac{1}{\Gamma(\tilde{\mu})} \left(\int_0^t \left[(t-s)^{\tilde{\mu}-1} - (t+\tau-s)^{\tilde{\mu}-1} \right] w(s)\, ds \right. \\
&\qquad\qquad\qquad \left. - \int_0^\tau (\tau - s)^{\tilde{\mu}-1} w(t+s)\, ds \right) \\
&\leq \frac{2\|w\|_{L^\infty(0,1)}}{\Gamma(\tilde{\mu}+1)} \tau^{\tilde{\mu}} \\
&= \frac{2\|w\|_{L^\infty(0,1)}}{\Gamma(\mu - m + 1)} \tau^{\mu - m} \qquad (50)
\end{aligned}
$$

exactly as in (48). (Again we note the constant 2 in (50) may be halved in the case $\mu = \nu + 1$.) Thus, in the case of $\mu \in (1, \nu + 1]$, $\mu \neq 2, 3, \ldots, \nu$, $\bar{u}^{(m)}$ is Hölder continuous on $[0, 1]$ with exponent $\mu - m$ and $h^{(m)} \in C[0, 1 - \alpha]$ for all $\alpha \in (0, \bar{\alpha}]$. Combining the above results it follows that for *all* $\mu \in (1, \nu + 1]$, the function h defined in (49) satisfies $h(\cdot; \alpha) \in C^m[0, 1 - \alpha]$, for $m = m(\mu)$.

We now return to estimates on $\|v_\alpha\|_{L^\infty(0,1-\alpha)}$ which we prove in two cases depending on the value of μ. We use the bound in (38), namely

$$\|v_\alpha\|_{L^\infty(0,1-\alpha)} \leq \hat{C}_\infty M_\infty(\alpha; h), \tag{51}$$

for $\hat{C}_\infty$ independent of α, where it remains to estimate $\left\|h^{(m)}\right\|_{L^\infty(0,1-\alpha)} / \alpha^{\nu - m}$ and $\left|h^{(j)}(0)\right| / \alpha^{\nu - j}$, $j = 0, 1, \ldots, m - 1$, in each case.

Case 1: Let $\mu \in (1, \nu + 1]$, for $\mu \neq 2, 3, \ldots, \nu$. Then $m = m(\mu) \in \{1, 2, \ldots, \nu\}$. For $j = 0, 1, \ldots, m$,

$$\bar{u}^{(j)}(t) = \frac{1}{\Gamma(\mu - j)} \int_0^t (t - s)^{\mu - j - 1} w(s)\, ds, \quad t \in [0, 1 - \alpha], \tag{52}$$

then for $j = 0, \ldots, m - 1$, it follows that $\bar{u}^{(j)}(0) = 0$,

$$\left|\bar{u}^{(j)}(t)\right| \leq \frac{\alpha^{\mu - j}}{\Gamma(\mu - j + 1)} \|w\|_\infty, \quad t \in [0, \alpha], \tag{53}$$

and thus

$$\frac{\left|h^{(j)}(0; \alpha)\right|}{\alpha^{\nu - j}} = \mathcal{O}\left(\frac{\alpha^{\nu + \mu - j}}{\alpha^{\nu - j}}\right) = \mathcal{O}\left(\alpha^\mu\right), \quad j = 0, 1, \ldots, m - 1. \tag{54}$$

Further, using (50),

$$\left\|\bar{u}^{(m)}(\cdot + \tau) - \bar{u}^{(m)}(\cdot)\right\|_{L^\infty(0,1-\alpha)} \leq \alpha^{\mu - m} \frac{2\|w\|_{L^\infty(0,1)}}{\Gamma(\mu - m + 1)},$$

for $\tau \in [0, \alpha]$, so that

$$\frac{\left\|h^{(m)}(\cdot; \alpha)\right\|_{L^\infty(0,1-\alpha)}}{\alpha^{\nu - m}} = \mathcal{O}\left(\frac{\alpha^{\nu + \mu - m}}{\alpha^{\nu - m}}\right) = \mathcal{O}\left(\alpha^\mu\right).$$

It follows then from (51) that $\|v_\alpha\|_\infty = \mathcal{O}\left(\alpha^\mu\right)$, as $\alpha \to 0$.

Case 2: Let $\mu \in \{2, \ldots, \nu\}$, so that $m(\mu) = \mu$. For $j = 0, \ldots, m - 1$, the estimates in (52)–(54) are unchanged. New, however, is the fact that

$$\bar{u}^{(m-1)}(t) = \frac{1}{\Gamma(1)} \int_0^t w(s)\, ds, \quad t \in [0, 1],$$

from which we find

$$\bar{u}^{(m)}(t) = w(t), \quad t \in [0, 1].$$

Thus,

$$\left|\bar{u}^{(m)}(t + \tau) - \bar{u}^{(m)}(t)\right| \leq 2\|w\|_{L^\infty(0,1)}$$

for $t \in [0, 1-\alpha]$, $\tau \in [0, \alpha]$, or

$$\frac{\|h^{(m)}\|_{L^\infty(0,1-\alpha)}}{\alpha^{\nu-m}} \leq \frac{2\alpha^\nu \|w\|_{L^\infty(0,1)}}{\alpha^{\nu-m}} = \mathcal{O}\left(\alpha^\mu\right).$$

Therefore $\|v_\alpha\|_{L^\infty(0,1-\alpha)} = \mathcal{O}\left(\alpha^\mu\right)$, as $\alpha \to 0$, in this case as well.

The desired rate of convergence in (46) is thus established. Then for some $\tilde{C}_\mu > 0$,

$$\|u_\alpha^\delta - r_\alpha \bar{u}\|_\alpha \leq C_1 \frac{\delta}{\alpha^\nu} + \tilde{C}_\mu \alpha^\mu,$$

so that if $\alpha = K\delta^{1/(\mu+\nu)}$ for some $K > 0$, then the convergence in (47) is obtained. □

4. Numerical Examples

The following examples are provided to briefly demonstrate the effectiveness of the generalized method of local regularization, and to illustrate the implication of Lemma 3.1 on the stable construction of the measure η_α used in the method. Additional examples for a variety of ν-smoothing Volterra problems with various relative data error may be found in [14, 15].

Let $N = 200$, $t_i = i/N$, $i = 1, \ldots, N$, and let $S_N = \text{span}\{\chi_i\}_{i=1}^N$, where for $i = 1, \ldots, N$, the indicator function χ_i is defined by $\chi_i(t) = 1$, $t \in (t_{i-1}, t_i]$, and $\chi_i(t) = 0$ otherwise. For each of the methods of local, Lavrent'ev, and Tikhonov regularization, we seek $u \in S_N$ which satisfies equation (10) at the collocation points t_i, $i = 1, \ldots, N$, where the values of a_α, A_α, and T_α in (10) are given by the particular method (see Remark 2.3).

All examples below pertain to the problem of approximating the true solution $\bar{u}$ given by

$$\bar{u}(t) = \begin{cases} -20t/3 + 1, & 0 \leq t \leq 0.3, \\ 5t - 5/2, & 0.3 < t \leq 0.5, \\ -5t + 5/2, & 0.5 < t \leq 0.7, \\ 20t/3 - 17/3, & 0.7 < t \leq 1, \end{cases} \tag{55}$$

which is shown as the dashed curve in each of the figures below. The exact data is given by $f = \mathcal{A}\bar{u}$, and at collocation points this data is represented by the vector $f_N = (f(t_1), \ldots, f(t_N))^\top \in \mathbb{R}^N$. A vector of uniformly distributed random error is added to f_N to create the noisy data vector f_N^δ with absolute error $\delta = \|f_N - f_N^\delta\|$ and relative data error given by $\|f_N - f_N^\delta\|/\|f_N\|$, where $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^N$. We will use the $u_{N,\alpha}^\delta \in \mathbb{R}^N$ to denote the vector with i^{th} component given by the collocation-based regularized solution u at $t = ((i - .5)/N)$ for given α, δ, and f_N^δ. To compare this approximation with the true solution $\bar{u}$ for each method, we compute the relative solution error, $\|\bar{u}_N - u_{N,\alpha}^\delta\|/\|\bar{u}_N\|$, where $\bar{u}_N \in \mathbb{R}^N$ is the vector with i^{th} component given by the value of $\bar{u}$ at $t = ((i - .5)/N)$, $i = 1, \ldots, N$.

The values of the regularization parameter for each method in the examples below are those for which the method's regularized solution is closest to the true solution (in a relative $\|\cdot\|$–norm sense) on either the entire interval $[0,1]$ (Example 4.1) or on the shortened interval $[0, 1-\alpha_{loc}]$ associated with local regularization (Examples 4.2–4.3). When the ideal solution $\bar{u}$ is not known, one would instead use a discrepancy principle for parameter selection; see [4] for a similar comparison of methods using such principles. Examples 4.1 and 4.2 involve a 1-smoothing problem with kernel $k(t) = e^{-t/2}$ and 3% relative error in the data. We compare the ability of the methods of Tikhonov, Lavrent'ev, and local regularization to reconstruct the three corners found in the true solution $\bar{u}$, which is generally impossible for all methods in the case of large relative errors, as well as the ability of the methods to approximate the flat areas in the solution.

Example 4.1. *In order to set a benchmark, we first illustrate the results found using Tikhonov regularization with the Tikhonov parameter α_{tik} chosen to minimize the relative solution error on $[0,1]$. The reconstruction uses $\alpha_{tik} = 2.92e^{-4}$ and has 17.1% relative error. See Figure 1 below.*

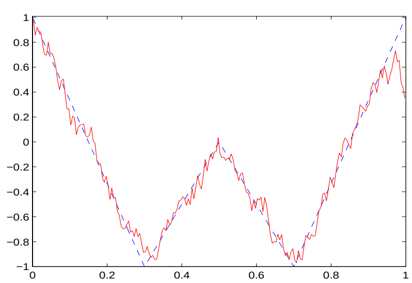

Figure 1. Optimal reconstruction of $\bar{u}$ on $[0,1]$ using Tikhonov regularization.

It is worth noting that in Example 4.1 Tikhonov regularization is unable to obtain a good construction of the solution near the end of the interval $[0,1]$, a phenomenon typical of Volterra (causal) problems regardless of the regularization method used. Thus while the generalized method of local regularization is disadvantaged because it only finds a reconstruction on the interval $[0, 1-\alpha_{loc}]$ for a given value of the local regularization parameter α_{loc}, the solution $\bar{u}$ generally cannot be well-approximated on the final interval $(1-\alpha_{loc}, 1]$ in any case. On the other hand, if sufficient data is available *beyond* the original interval, then both Tikhonov regularization and local regularization can be expected to do well on all of $[0,1]$.

Example 4.2. *Using the local regularization parameter $\alpha_{loc} = .08$, we obtain the reconstruction shown on the left in Figure 2, which has a relative solution error on the interval $[0, .92]$ of 10.5%. We also compare reconstructions obtained using the methods of Tikhonov and Lavrent'ev with the locally-regularized solution, selecting the associated regularization parameters α_{tik} and α_{lav} for these two methods such that relative solution error is minimized on the same interval $[0, .92]$ as that used for local regularization. (Note however that we still graph the reconstructed solution for each of these two methods on the entire interval $[0,1]$.)*

The result for Tikhonov regularization is shown on the right in Figure 2, and the relative solution error in this case reduces to 10.0%. *Despite the smaller error, it can be observed that the Tikhonov solution is overly smooth and unable to recover the "corners" in $\bar{u}$ as effectively as the method of local regularization, while the oversmoothing actually benefits the Tikhonov solution on the flat areas. In making this comparison one should also recall that Tikhonov regularization comes with increased computational costs; that is, the discretized Tikhonov algorithm requires the solution of a full matrix equation, in contrast to the more efficient forward elimination algorithm which can be used to solve the lower triangular matrix equation associated with either local regularization or the method of Lavrent'ev.*

Figure 3 shows the "optimal" reconstruction of the solution using Lavrent'ev regularization, with relative solution error 35.3%. *Despite the poor reconstruction, the method of Lavrent'ev is known to converge for a one-smoothing Volterra equation in which the kernel is nonnegative, nonincreasing, and convex [11, 20].*

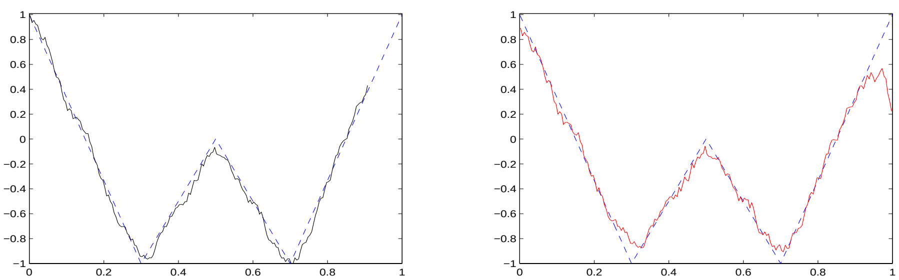

Figure 2. *Graph on left:* Construction of the solution using local regularization with regularization parameter $\alpha_{loc} = 0.08$, with relative solution error 10.5%.
Graph on right: Optimal reconstruction on $[0, 0.92]$ using Tikhonov regularization ($\alpha_{tik} = 5.58e^{-4}$), with relative solution error 10.0%.

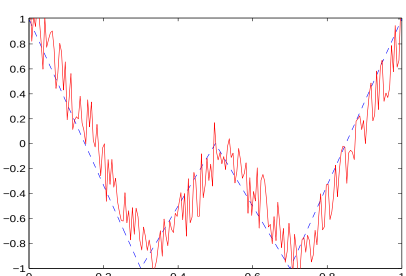

Figure 3. Optimal reconstruction on $[0, 0.92]$ using the method of Lavrent'ev ($\alpha_{lav} = 2.59e^{-2}$), with relative solution error 35.3%.

In Example 4.3, we illustrate how to construct a suitable measure η_α for the 3-smoothing problem in which $k(t) = t^2/2$ with 0.1% relative data error, and discuss related stability issues of this construction.

Example 4.3. *Following the proof of Proposition 3.2, the steps for construction of an* η_α *satisfying Definition 3.1 are straightforward:*

(i) We first select a 3rd degree polynomial p_3 *for which the roots all have negative real part, e.g.,* $p_3(\lambda) = (\lambda+5)^3$. *Expanding, we have*

$$p_3(\lambda) = \lambda^3 + 15\lambda^2 + 75\lambda + 125,$$

so that the scalars defined in the representation (25) for p_3 *are given by* $c_0 = 125,\ c_1 = 15,\ c_2 = 30,$ *and* $c_3 = 6$.

(ii) Given p_3 *define* η_α *using (27), where the polynomial* $\psi(\rho) = \sum_{i=0}^{3} d_i \rho^i$ *uniquely satisfies*

$$\int_0^\alpha \rho^j d\eta_\alpha(\rho) = \int_0^\alpha \rho^j \psi\left(\frac{\rho}{\alpha}\right) d\rho = \alpha^{1+j} \int_0^1 \rho^j \psi(\rho)\, d\rho = \alpha^{\sigma+j} c_j,$$

for $j = 0, 1, 2, 3,$ *with* $\sigma = 1$. *Then since*

$$\int_0^1 \rho^j \psi(\rho)\, d\rho = \sum_{i=0}^{3} d_i \int_0^1 \rho^{i+j} d\rho = \sum_{i=0}^{3} d_i \left(\frac{1}{i+j+1}\right),$$

the coefficients d_i *in* ψ *satisfy*

$$\sum_{i=0}^{3} d_i \left(\frac{1}{i+j+1}\right) = c_j, \quad j = 0, 1, 2, 3, \tag{56}$$

leading to

$$\psi(\rho) = -700\rho^3 - 3300\rho^2 + 4080\rho - 640. \tag{57}$$

(iii) A suitable measure η_α *for the 3-smoothing problem is then given by (27) using the polynomial* ψ *in (57).*

Due to the fact that the 4×4 *linear system in (56) is ill-conditioned, it can be beneficial to apply a small amount of stabilization when solving (56) for the coefficients* d_i *in* ψ. *Following the approach detailed in Lemma 3.1, we solve (56) using Tikhonov regularization with a small Tikhonov parameter* $\beta = 0.0001$ *resulting in the stabilized polynomial* $\tilde{\psi}$,

$$\tilde{\psi}(\rho) = -837.3113\rho^3 - 171.8107\rho^2 + 893.4561\rho - 51.9243, \tag{58}$$

and the stabilized measure $\tilde{\eta}_\alpha$ *constructed from* $\tilde{\psi}$ *via (27). In order to use this new measure with local regularization, we first need to check that the conditions in Definition 3.1 are still satisfied. Indeed, in the verification of (i)(b) in Definition 3.1, the polynomial* p_3 *in (25) is given by*

$$p_3(\lambda) = 2.9099\lambda^3 + 16.0709\lambda^2 + 61.4416\lambda + 128.2056,$$

which has roots given (to 4 decimal places) by $-1.1531 \pm 3.5167i$, *and* -3.2168. *Thus the stabilized measure* $\tilde{\eta}_\alpha$ *is indeed local-regularizing.*

The local-regularized reconstruction shown on the left in Figure 4 makes use of the unstabilized measure η_α *which comes from the direct (unregularized) solution of (56).*

The local-regularized reconstruction on the right in Figure 4 uses the stabilized measure $\tilde{\eta}_\alpha$ from the regularized solution of (56). The relative solution error for the former reconstruction is 39.3%*, compared to* 8.8% *for the reconstruction using the stabilized measure.*

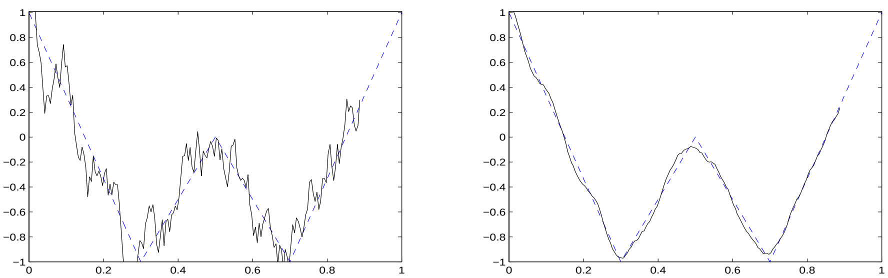

Figure 4. *Graph on left:* Construction of the solution using local regularization with regularization parameter $\alpha_{loc} = 0.1$ and unstable measure η_α. Relative solution error is 39.3% *Graph on right:* Construction of the solution using local regularization with regularization parameter $\alpha_{loc} = 0.1$ and stable measure $\tilde{\eta}_\alpha$. Relative solution error is 8.8%

5. Acknowledgments

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6. Appendix: Proof of Lemma 3.3

Proof. Fix $m \in \{1, \ldots, \nu\}$ and $\bar{\alpha} > 0$ so that the bounds in Lemma 3.2 hold. For every $\alpha \in (0, \bar{\alpha}]$, equation (37) is well-posed for $h \in C^m[0, 1-\alpha]$ and has a unique solution in $C[0, 1-\alpha]$ depending continuously on $h \in C[0, 1-\alpha]$. Thus it remains to establish the bound in (38).

We make use of Laplace transform methods for the analysis of equation (37) requiring that extensions of h and k_α to $[0, \infty)$ be defined. To this end, let $g_h \in C[0, \infty)$ be any continuous extension of $h^{(m)} \in C[0, 1-\alpha]$ satisfying $g_h(t) = h^{(m)}(t)$, $t \in [0, 1-\alpha]$, $g_h(t) = 0$ for $t \geq 2$, and $|g_h(t)| \leq \sup_{0 \leq s \leq 1-\alpha} |h^{(m)}(s)|$, $t \in [0, \infty)$. Define the extension of h to $[0, \infty)$ as the unique solution to the initial value problem

$$\begin{aligned} &v^{(m)}(t) = g_h(t), \quad t \in [0, \infty), \\ &v(0) = h(0),\ v'(0) = h'(0),\ \ldots,\ v^{(m-1)}(0) = h^{(m-1)}(0). \end{aligned}$$

Similarly extend the function $k_\alpha \in C^\nu[0, 1-\alpha]$ as the unique solution to the initial value problem

$$\begin{aligned} &v^{(m)}(t) = g_\alpha(t), \quad t \in [0, \infty), \\ &v(0) = k_\alpha(0),\ v'(0) = k'_\alpha(0),\ \ldots,\ v^{(m-1)}(0) = k_\alpha^{(m-1)}(0), \end{aligned}$$

using any $g_\alpha \in C[0,\infty)$ satisfying $g_\alpha(t) = k_\alpha^{(\nu)}(t)$, $t \in [0, 1-\alpha]$, $g_\alpha(t) = 0$ for $t \geq 2$, and

$$|g_\alpha(t)| \leq \max_{0 \leq s \leq 1-\alpha} |k_\alpha^{(\nu)}(s)| \leq M_T \left\| k^{(\nu)} \right\|_{C[0,1]}, \quad t \in [0,\infty). \tag{59}$$

For arbitrary $\alpha \in (0, \bar{\alpha}]$, note that the solution y_α of (37) satisfies

$$y(\alpha t) + \int_0^{\alpha t} \frac{k_\alpha(\alpha t - s)}{a_\alpha} y(s)\, ds = \frac{h(\alpha t)}{a_\alpha}, \quad t \in [0, (1-\alpha)/\alpha].$$

If we make a change of variables and define

$$\hat{y}_\alpha(t) := \alpha y_\alpha(\alpha t), \quad t \in [0, (1-\alpha)/\alpha], \tag{60}$$

it follows that

$$\hat{y}_\alpha(t) + \int_0^t \alpha \frac{k_\alpha(\alpha(t-s))}{a_\alpha} \hat{y}_\alpha(s) ds = \alpha \frac{h(\alpha t)}{a_\alpha}, \tag{61}$$

for all $t \in [0, (1-\alpha)/\alpha]$. Extend this equation to $t \in [0, \infty)$ (using the extensions k_α, $h \in L^1(0,\infty)$ defined earlier). Then there is a unique extension of $\hat{y}_\alpha$ (again called $\hat{y}_\alpha$) in $L^1(0,\infty)$ which satisfies (61) for a.e. $t \in (0,\infty)$ [9].

Apply the Laplace transform to both sides of equation (61) on $[0,\infty)$ to obtain

$$\mathcal{L}\{\hat{y}_\alpha\} + \frac{\alpha}{a_\alpha} \mathcal{L}\{k_\alpha(\alpha\,\cdot)\} \mathcal{L}\{\hat{y}_\alpha\} = \frac{\alpha}{a_\alpha} \mathcal{L}\{h(\alpha\,\cdot)\}. \tag{62}$$

It follows from use of the Laplace transform identity,

$$s^r \mathcal{L}\{w\}(s) = \sum_{\ell=0}^{r-1} w^{(r-1-\ell)}(0)\, s^\ell + \mathcal{L}\{w^{(r)}\}(s),$$

for sufficiently smooth $w : [0,\infty) \mapsto \mathbb{R}$ and $r = 1, 2, \ldots$, that

$$s^\nu \mathcal{L}\{k_\alpha(\alpha\,\cdot)\}(s) = \sum_{\ell=0}^{\nu-1} \alpha^{\nu-1-\ell} k_\alpha^{(\nu-1-\ell)}(0)\, s^\ell + \alpha^\nu \mathcal{L}\{k_\alpha^{(\nu)}(\alpha\,\cdot)\}(s), \tag{63}$$

for $s \in \mathbb{C}$. Further, using the Taylor expansion of $k^{(i)}(t)$ at 0 for each $i = 0, 1, \ldots, \nu-1$ and Definition 3.1, we obtain

$$\begin{aligned} k_\alpha^{(i)}(0) &= \frac{1}{\gamma_\alpha} \left[\int_0^\alpha \frac{\rho^{\nu-1-i}}{(\nu-1-i)!} d\eta_\alpha(\rho) + \int_0^\alpha k^{(\nu)}(\xi_{\rho,\nu-1-i}) \frac{\rho^{\nu-i}}{(\nu-i)!} d\eta_\alpha(\rho) \right] \\ &= \frac{1}{\gamma_\alpha} \left[\frac{c_{\nu-1-i}}{(\nu-1-i)!} \alpha^{\nu-1-i+\sigma} \left(1 + C_{\nu-1-i}(\alpha)\right) + \int_0^\alpha k^{(\nu)}(\xi_{\rho,\nu-1-i}) \frac{\rho^{\nu-i}}{(\nu-i)!} d\eta_\alpha(\rho) \right], \end{aligned}$$

for $i = 0, 1, \ldots, \nu-1$. Division by the expansion of a_α in (34) yields

$$\frac{\alpha^{i+1}}{a_\alpha} k_\alpha^{(i)}(0) = \frac{c_{\nu-1-i}}{(\nu-1-i)!} + \bar{m}_{\nu-1-i,\alpha}, \quad i = 0, 1, \ldots, \nu-1, \tag{64}$$

for the quantity $\bar{m}_{j,\alpha}$ defined by

$$\bar{m}_{j,\alpha} := T_{1,j}(\alpha) + T_{2,j}(\alpha), \quad j = 0, 1, \ldots, \nu-1,$$

where

$$T_{1,j}(\alpha) := \frac{1}{a_\alpha \gamma_\alpha} \left[\alpha^{\nu-j} \int_0^\alpha k^{(\nu)}(\xi_{\rho,j}) \frac{\rho^{j+1}}{(j+1)!} d\eta_\alpha(\rho) - \frac{c_j}{j!} \int_0^\alpha \int_0^\rho k^{(\nu)}(\zeta_\rho) \frac{s^\nu}{\nu!} ds d\eta_\alpha(\rho) \right]$$

and

$$T_{2,j}(\alpha) := \frac{1}{a_\alpha \gamma_\alpha} \frac{c_j}{j!} \alpha^{\nu+\sigma} \left(C_j(\alpha) - C_\nu(\alpha) \right).$$

It follows from (63) and (64) that, for $s \in \mathbb{C}$,

$$s^\nu \frac{\alpha}{a_\alpha} \mathcal{L}\{k_\alpha(\alpha\,\cdot\,)\}(s) = \sum_{\ell=0}^{\nu-1} \left(\frac{c_\ell}{\ell!} + \bar{m}_{\ell,\alpha} \right) s^\ell + \frac{\alpha^{\nu+1}}{a_\alpha} \mathcal{L}\{k_\alpha^{(\nu)}(\alpha\,\cdot\,)\}(s),$$

and

$$s^m \frac{\alpha}{a_\alpha} \mathcal{L}\{h(\alpha\,\cdot\,)\}(s) = \sum_{\ell=0}^{m-1} \frac{\alpha^{m-\ell}}{a_\alpha} h^{(m-1-\ell)}(0)\, s^\ell + \frac{\alpha^{m+1}}{a_\alpha} \mathcal{L}\{h^{(m)}(\alpha\,\cdot\,)\}(s).$$

Define the polynomial $p = p(\cdot;\alpha)$ via

$$p(s) := s^\nu + \sum_{\ell=0}^{\nu-1} \left(\frac{c_\ell}{\ell!} + \bar{m}_{\ell,\alpha} \right) s^\ell = \sum_{\ell=0}^{\nu} \left(\frac{c_\ell}{\ell!} + \bar{m}_{\ell,\alpha} \right) s^\ell, \tag{65}$$

$s \in \mathbb{C}$, where in the second equality, we define $\bar{m}_{\nu,\alpha} := 0$ and use the fact that $c_\nu/\nu! = 1$. Then multiplying (62) through by s^ν,

$$\begin{aligned} &p(s)\mathcal{L}\{\hat{y}_\alpha\}(s) + \frac{\alpha^{\nu+1}}{a_\alpha} \mathcal{L}\{k_\alpha^{(\nu)}(\alpha\,\cdot\,)\}(s)\mathcal{L}\{\hat{y}_\alpha\}(s) \\ &= \sum_{\ell=0}^{m-1} \frac{\alpha^{m-\ell}}{a_\alpha} h^{(m-1-\ell)}(0)\, s^{\nu-m+\ell} + s^{\nu-m} \frac{\alpha^{m+1}}{a_\alpha} \mathcal{L}\{h^{(m)}(\alpha\,\cdot\,)\}(s), \end{aligned} \tag{66}$$

for $s \in \mathbb{C}$. But the polynomial $p(\cdot;\alpha)$ in (65) is a perturbation of the polynomial p_ν defined in (25) for which the roots all have negative real part. Thus if we show that $|\bar{m}_{\ell,\alpha}|$ is sufficiently small for $\ell = 0, 1, \ldots, \nu-1$, then p may be written

$$p(s) = (s-\lambda_1)(s-\lambda_2)\cdots(s-\lambda_\nu),$$

where the real part of $\lambda_n = \lambda_n(\alpha)$ is negative for each $n = 1, \ldots, \nu$.

In order to estimate $|\bar{m}_{j,\alpha}|$, use Lemma 3.2 to obtain

$$T_{1,j}(\alpha) \le \frac{\bar{\alpha} M_T \left\| k^{(\nu)} \right\|_{C[0,1]}}{\kappa_1} \left[\frac{1}{(j+1)!} + \frac{|c_j|}{j!(\nu+1)!} \right]$$

and

$$T_{2,j}(\alpha) \le \bar{\alpha} \frac{1}{\kappa_1 c_0 \left(1 - \bar{C}_0 \bar{\alpha}\right)} \frac{|c_j|}{j!} \left(\bar{C}_j + \bar{C}_\nu \right),$$

so that

$$|\bar{m}_{j,\alpha}| \le \xi_j \bar{\alpha}, \tag{67}$$

with the $\xi_j > 0$ constants independent of α, for all $\alpha \in (0, \bar{\alpha}]$ and $j = 0, 1, \ldots, \nu-1$. Thus, for $\bar{\alpha} > 0$ sufficiently small, the roots of $p(\cdot;\alpha)$ have negative real part for all $\alpha \in (0, \bar{\alpha}]$.

Returning to equation (66) and dividing through by $p(s)$,

$$\begin{aligned}\mathcal{L}\{\hat{y}_\alpha\}(s) + v_0(s)\frac{\alpha^{\nu+1}}{a_\alpha}\mathcal{L}\{k_\alpha^{(\nu)}(\alpha\,\cdot)\}(s)\mathcal{L}\{\hat{y}_\alpha\}(s) \\ = \sum_{\ell=0}^{m-1}\frac{\alpha^{m-\ell}}{a_\alpha}h^{(m-1-\ell)}(0)\,v_{\nu-m+\ell}(s) + v_{\nu-m}(s)\frac{\alpha^{m+1}}{a_\alpha}\mathcal{L}\{h^{(m)}(\alpha\,\cdot)\}(s),\end{aligned} \tag{68}$$

where for $j = 0$ and $j = \nu - m, \ldots \nu - 1$, each v_j given by

$$v_j = \frac{s^j}{p(s)},$$

is a proper rational function. Using partial fractions to rewrite v_j, it is easy to see that the inverse Laplace transform $V_j := \mathcal{L}^{-1}\{v_j\}$ of v_j is a linear combination of functions of the form $t^\mu e^{\lambda_\ell t}$ (for $\lambda_\ell < 0$ real) and $e^{a_\ell t}\, t^\mu \cos b_\ell t$ (for $\lambda_\ell = a_\ell + b_\ell i$, $a_\ell < 0$, $b_\ell \neq 0$, and some $\mu \in \{1, \ldots, \nu - 1\}$); that is, $V_j \in L^p(0, \infty)$, for $j = 0$ and $j = \nu - m, \ldots, \nu - 1$ and all $1 \le p \le \infty$.

Next we apply the inverse Laplace transform to both sides of equation (68) and restrict the resulting equation to $t \in [0, (1-\alpha)/\alpha]$, leading to

$$\hat{y}_\alpha(t) + \int_0^t K_\alpha(t-s)\hat{y}_\alpha(s)\,ds = H_\alpha(t), \;\; t \in [0, (1-\alpha)/\alpha], \tag{69}$$

where

$$K_\alpha(t) = \frac{\alpha^{\nu+1}}{a_\alpha}\int_0^t V_0(t-\tau)k_\alpha^{(\nu)}(\alpha\tau)\,d\tau,$$

and

$$\begin{aligned}H_\alpha(t) = \sum_{\ell=0}^{m-1}\frac{\alpha^{m-\ell}}{a_\alpha}h^{(m-1-\ell)}(0)\,V_{\nu-m+\ell}(t) \\ + \frac{\alpha^{m+1}}{a_\alpha}\int_0^t V_{\nu-m}(t-\tau)h^{(m)}(\alpha\tau)\,d\tau,\end{aligned}$$

for $t \in [0, (1-\alpha)/\alpha]$.

We return now to the original variable y_α via the relationship in (60), make this substitution into equation (69) along with the substitution t for αt, and obtain

$$\alpha y_\alpha(t) + \int_0^{t/\alpha} K_\alpha((t/\alpha) - s)\,\alpha y_\alpha(\alpha s)\,ds = H_\alpha(t/\alpha), \;\; t \in [0, 1-\alpha].$$

A further change of integration variable leads to the equation

$$y_\alpha(t) + \frac{1}{\alpha}\int_0^t K_\alpha((t-s)/\alpha)\,y_\alpha(s)\,ds = \frac{1}{\alpha}H_\alpha(t/\alpha), \;\; t \in [0, 1-\alpha]. \tag{70}$$

First note that, $\left\|V_0\left(\frac{\cdot}{\alpha}\right)\right\|_\alpha \le \alpha^{1/p}\,\|V_0\|_{L^p(0,\infty)}$, and

$$\left\|\frac{1}{\alpha}H_\alpha\left(\frac{\cdot}{\alpha}\right)\right\|_\alpha = \alpha^{1/p}\left\|\frac{1}{\alpha}H_\alpha(\cdot)\right\|_{L^p(0,(1-\alpha)/\alpha)}, \tag{71}$$

(with $1/p = 0$ in the case of $p = \infty$). Thus, for $t \in [0, 1-\alpha]$,

$$
\begin{aligned}
\frac{1}{\alpha}|K_\alpha(t/\alpha)| &\le \frac{\alpha^\nu}{a_\alpha}\int_0^{t/\alpha} |V_0((t/\alpha)-\tau)| \left|k_\alpha^{(\nu)}(\alpha\tau)\right| d\tau \\
&= \frac{\alpha^{\nu-1}}{a_\alpha}\int_0^{t} |V_0((t-\tau)/\alpha)| \left|k_\alpha^{(\nu)}(\tau)\right| d\tau \\
&\le \frac{1}{\kappa_1\alpha}\left\|V_0(\cdot/\alpha)\right\|_{L^1(0,1-\alpha)} \left\|k_\alpha^{(\nu)}\right\|_{L^\infty(0,1-\alpha)},
\end{aligned}
$$

so that

$$
\left\|\frac{1}{\alpha}K_\alpha\left(\frac{\cdot}{\alpha}\right)\right\|_{L^\infty(0,1-\alpha)} \le C_K,
$$

where the scalar $C_K = \dfrac{M_T}{\kappa_1}\left\|V_0\right\|_{L^1(0,\infty)}\left\|k^{(\nu)}\right\|_{L^\infty(0,1)}$ is independent of α.

We also have

$$
\begin{aligned}
&\left\|\frac{1}{\alpha}H_\alpha\left(\cdot\right)\right\|_{L^p(0,(1-\alpha)/\alpha)} \\
&\le \sum_{j=0}^{m-1}\frac{|h^{(j)}(0)|}{\kappa_1\alpha^{\nu-j}}\|V_{\nu-j-1}\|_{L^p(0,\infty)} + \frac{1}{\kappa_1\alpha^{\nu-m}}\|V_{\nu-m}(\cdot)\star h^{(m)}(\alpha\cdot)\|_{L^p(0,(1-\alpha)/\alpha)}
\end{aligned}
$$

and using Young's inequality for convolutions, for $1 \le p < \infty$,

$$
\|V_{\nu-m}(\cdot)\star h^{(m)}(\alpha\cdot)\|_{L^p(0,(1-\alpha)/\alpha)} \le \frac{1}{\alpha}\left\|h^{(m)}\right\|_{L^\infty(0,1-\alpha)}\|V_{\nu-m}\|_{L^p(0,\infty)}
$$

and

$$
\|V_{\nu-m}(\cdot)\star h^{(m)}(\alpha\cdot)\|_{L^\infty(0,(1-\alpha)/\alpha)} \le \left\|h^{(m)}\right\|_{L^\infty(0,1-\alpha)}\|V_{\nu-m}\|_{L^1(0,\infty)}.
$$

Therefore with (71),

$$
\left\|\frac{1}{\alpha}H_\alpha\left(\frac{\cdot}{\alpha}\right)\right\|_\alpha \le \hat{C}_p\, M_p(\alpha;h)
$$

where, for $1 \le p < \infty$, $M_p(\alpha;h)$ and $M_\infty(\alpha;h)$ are given by the statement of the theorem, and the scalars $\hat{C}_p$ are independent of α for all $1 \le p \le \infty$.

Returning to equation (70), it follows that

$$
|y_\alpha(t)| \le C_K\int_0^t |y_\alpha(s)|\,ds + \frac{1}{\alpha}\left|H_\alpha\left(t/\alpha\right)\right|, \quad t \in [0,1-\alpha],
$$

or, using Gronwall's inequality,

$$
|y_\alpha(t)| \le \frac{1}{\alpha}\left|H_\alpha\left(t/\alpha\right)\right| + C_K\int_0^t \frac{1}{\alpha}\left|H_\alpha\left(s/\alpha\right)\right|\exp(C_K(t-s))\,ds,
$$

for $t \in [0,1-\alpha]$. Thus, for $1 \le p \le \infty$,

$$
\begin{aligned}
\|y_\alpha\|_\alpha &\le \left\|\frac{1}{\alpha}H_\alpha\left(\frac{\cdot}{\alpha}\right)\right\|_\alpha + C_K\left\|\frac{1}{\alpha}H_\alpha\left(\frac{\cdot}{\alpha}\right)\right\|_\alpha \|\exp(C_K\,\cdot\,)\|_{L^1(0,1-\alpha)} \\
&\le \hat{C}_p\exp(C_K)M_p(\alpha;h),
\end{aligned}
$$

from which the bounds in (38) are found. □

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